## 10 GEOMETRIC DISTRIBUTION

## EXAMPLES:

1. Terminals on an on-line computer system are attached to a communication line to the central computer system. The probability that any terminal is ready to transmit is 0.95 .
Let $\mathrm{X}=$ number of terminals polled until the first ready terminal is located.
2. Toss a coin repeatedly.

Let $\mathrm{X}=$ number of tosses to first head
3. It is known that $20 \%$ of products on a production line are defective. Products are inspected until first defective is encountered.
Let $\mathrm{X}=$ number of inspections to obtain first defective
4. One percent of bits transmitted through a digital transmission are received in error. Bits are transmitted until the first error.
Let $X$ denote the number of bits transmitted until the first error.

## GEOMETRIC DISTRIBUTION

## Conditions:

1. An experiment consists of repeating trials until first success.
2. Each trial has two possible outcomes;
(a) A success with probability $p$
(b) A failure with probability $q=1-p$.
3. Repeated trials are independent.
$\mathrm{X}=$ number of trials to first success

X is a GEOMETRIC RANDOM VARIABLE.

## PDF:

$$
P(X=x)=q^{x-1} p ; \quad x=1,2,3, \cdots
$$

CDF:

$$
\begin{aligned}
P(X \leq x) & =P(X=1)+P(X=2) \cdots P(X=x) \\
& =p+q p+q^{2} p \cdots+q^{x-1} p \\
& =p\left[1-q^{x}\right] /(1-q) \\
& =1-q^{x}
\end{aligned}
$$

## Example:

Products produced by a machine has a $3 \%$ defective rate.

- What is the probability that the first defective occurs in the fifth item inspected?

$$
\begin{aligned}
P(X=5) & =P(1 \text { st } 4 \text { non-defective }) P(5 \text { th defective }) \\
& =\left(0.97^{4}\right)(0.03)
\end{aligned}
$$

In $R$
>dgeom ( $\mathrm{x}=4$, prob $=.03$ )
[1] 0.02655878
The convention in $R$ is to record $X$ as the number of failures that occur before the first success.

- What is the probability that the first defective occurs in the first five inspections?

$$
\begin{aligned}
& \qquad \begin{aligned}
& P(X \leq 5)=1-P(\text { First } 5 \text { non-defective }) \\
&=1-0.97^{5} \\
&>\operatorname{pgeom}(4, .03)
\end{aligned} \\
& \text { [1] } 0.1412660
\end{aligned}
$$

## Geometric $p d f s$

First Ready Terminal, $\mathrm{p}=.95$


First Defective, p=. 2



First Bit in Error, p=. 01


Calculating pdfs in $R$

```
par (mfrow = c(2,2))
x<-0:4
plot(x+1, dgeom(x, prob = .95),
    xlab = "X = Number of Trials", ylab = "P(X=x)",
    type = "h", main = "First Ready Terminal, p = .95")
x<-0:9
plot(x+1, dgeom(x, prob = .5),
    xlab = "X = Number of Trials", ylab = "P(X=x)",
        type = "h", main = "First Head, p = .5")
x<- 0:19
plot(x+1, dgeom(x, prob = .2),
    xlab = "X = Number of Trials", ylab = "P(X=x)",
    type = "h", main = "First Defective, p = .2")
x<- seq(0, 400, 50)
plot(x+1, dgeom(x, prob = .01),
    xlab = "X = Number of Trials", ylab = "P(X=x)",
    type = "h", main = "First Bit in Error, p = .01")
```

First Ready Terminal, $\mathbf{p}=.9$


First Defective, p=. 2

$X=$ Number of Trials

First Bit in Error, p = . 01


Figure 1: Geometric cdfs

```
par (mfrow = c(2,2))
x<-0:4
plot(x+1, pgeom(x, prob = .95),
    xlab = "X = Number of Trials", ylab = "P(X<=x)",
    type = "s", main = "First Ready Terminal, p = .95")
x<-0:9
plot(x+1, pgeom(x, prob = .5),
    xlab = "X = Number of Trials", ylab = "P(X<=x)",
    type = "s", main = "First Head, p = .5")
x<-0:19
plot(x+1, pgeom(x, prob = .2),
    xlab = "X = Number of Trials", ylab = "P(X< =x)",
    type = "s", main = "First Defective, p = .2")
x<- seq(0, 399)
plot(x+1, pgeom(x, prob = .01),
    xlab = "X = Number of Trials", ylab = "P(X<=x)",
    type = "s", main = "First Bit in Error, p = .01")
```


## The Quantile Function

In Example 3, a production line which has a $20 \%$ defective rate, what is the minimum number of inspections, that would be necessary so that the probability of observing a defective is more that $75 \%$ ?

Choose $k$ so that

$$
P(X \leq k) \geq .75 .
$$

In $R$
qgeom(.75, .2)
[1] 6
i.e. 6 failures before first success.
or with 7 inspections, there is at least a $75 \%$ chance of obtaining the first defective.

## Mean of geometric distribution:

## Example:

If a production line has a $20 \%$ defective rate. What is the average number of inspections to obtain the first defective?

$$
\begin{aligned}
E(X) & =\sum_{x=1}^{\infty} x q^{x-1} p \\
& =p \sum_{x=1}^{\infty} x q^{x-1} \\
& =p \sum_{x=1}^{\infty} \frac{d q^{x}}{d q} \\
& =p \frac{d \sum_{x=1}^{\infty} q^{x}}{d q} \\
& =p \frac{d(q /(1-q)}{d q} \\
& =p \frac{[(1-q)+q]}{(1-q)^{2}} \\
& =\frac{p}{p^{2}}=\frac{1}{p}
\end{aligned}
$$

Average number of inspections to obtain the first defective:

$$
E(X)=\frac{1}{.2}=5
$$

## The Markov Property:

If the probability of events happening in the future is independent of what went before, then the random variable is said to have the Markov property.

## MARKOV PROPERTY <br> $\Longrightarrow$ MEMORYLESS PROPERTY

## Example:

Products are inspected until first defective is found. $X$ is a geometric random variable with parameter $p$. The first 10 trials have been found to be free of defectives. What is the probability that the first defective will occur in the 15th trial?

Let $E_{1}$ be the event that first ten trials are free of defectives.
Let $E_{2}$ be the event that that first defective will occur on the 15th trial.

$$
\begin{aligned}
P(X=15 \mid X>10) & =P\left(E_{2} \mid E_{1}\right) \\
& =\frac{P\left(E_{1} \cap E_{2}\right)}{P\left(E_{1}\right)} \\
& =\frac{P(X=15 \cap X>10)}{P(X>10)} \\
& =\frac{P(X=15)}{P(X>10)} \\
& =\frac{q^{14} p}{q^{10}}=q^{4} p=P(X=5)
\end{aligned}
$$

## MARKOV PROPERTY

Generally, the Markov property states:

$$
P(X=x+n \mid X>n)=P(X=x)
$$

## Proof:

Let

$$
\begin{aligned}
& E_{1}=\{X>n\} \\
& E_{2}=\{X=x+n\}
\end{aligned}
$$

Then we may write

$$
P(X=x+n \mid X>n)=P\left(E_{2} \mid E_{1}\right)
$$

But

$$
P\left(E_{2} \mid E_{1}\right)=\frac{P\left(E_{1} \cap E_{2}\right)}{P\left(E_{1}\right)}
$$

Now

$$
P\left(E_{1} \cap E_{2}\right)=P(X=x+n)=q^{x+n-1} p
$$

And

$$
P\left(E_{1}\right)=P(X>n)=q^{n}
$$

Thus

$$
\begin{aligned}
P\left(E_{2} \mid E_{1}\right) & =\frac{q^{x+n-1} p}{q^{n}} \\
& =q^{x-1} p
\end{aligned}
$$

But

$$
P(X=x)=q^{x-1} p
$$

Hence

$$
P(X=x+n \mid(X>n)=P(X=x)
$$

## $\boldsymbol{R}$ Functions for the Geometric Distribution

- dgeom
dgeom ( $\mathrm{x}=4$, prob $=.03$ )
the probability of
exactly 4 trials before first defective or exactly 5 trials to first defective
- pgeom
pgeom ( $\mathrm{x}=4, \mathrm{prob}=.03$ )
the probability of
up to 4 trials before first defective or
up to 5 trials to first defective
- qgeom
qgeom(.75, .2)
returns the number of trials before first defective that has a probabilty of .75 .

