

FUNCTIONS

ABSOLUTE VALUE & INEQUALITIES

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Outline

- 1 Inequalities involving the Absolute Value
- 2 Solving More General Inequalities

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Absolute Value & Inequalities

Illustration 1

The equation

$$|2x + 1| = 3$$

has two solutions, obtained from solving

$$2x + 1 = +3 \quad \text{and} \quad 2x + 1 = -3$$

or  $x = 1$  and  $x = -2$ . Therefore, the solution to the absolute value equation  $|2x + 1| = 3$  is the set:

$$\{-2, 1\}$$

## Absolute Value & Inequalities

### Illustration 2

Consider the equation

$$|x| \geq 2$$

Either  $x \geq 2$  or  $x \leq -2$  and the solution may be expressed in interval form as

$$(-\infty, -2] \cup [2, \infty)$$

## Absolute Value & Inequalities

Solve the inequality  $|x^2 - 4| > 3$ .

The solutions are found from either:

$$\begin{array}{ccc} x^2 - 4 > 3 & \text{or} & x^2 - 4 < -3 \\ \downarrow & & \downarrow \\ x^2 > 7 & & x^2 < 1 \\ |x| > \sqrt{7} & & |x| < 1 \end{array}$$

giving 4 cases:

$$x > \sqrt{7}, \quad x < -\sqrt{7}, \quad x < 1, \quad x > -1.$$

which can be written in interval form as follows:

$$x \in (-\infty, -\sqrt{7}) \cup (-1, 1) \cup (\sqrt{7}, \infty)$$

## Absolute Value & Inequalities

Find all the real roots of the equation

$$|x + 1| + |2x - 3| = 4.$$

Since either  $|x + 1| = x + 1$  or  $-(x + 1)$  and  $|2x - 3| = 2x - 3$  or  $-(2x - 3)$ , there are 4 cases to consider:

$$\begin{array}{ll} +(x + 1) + (2x - 3) = 4 & \Rightarrow 3x - 2 = 4 \Rightarrow x = 2 \\ -(x + 1) + (2x - 3) = 4 & \Rightarrow x - 4 = 4 \Rightarrow x = 8 \\ +(x + 1) - (2x - 3) = 4 & \Rightarrow -x + 4 = 5 \Rightarrow x = 0 \\ -(x + 1) - (2x - 3) = 4 & \Rightarrow -3x + 2 = 4 \Rightarrow x = -\frac{2}{3} \end{array}$$

The **possible** solutions are therefore:

$$\left\{ 2, 8, 0, -\frac{2}{3} \right\}$$

## Absolute Value & Inequalities

Possible solutions:  $\left\{ 2, 8, 0, -\frac{2}{3} \right\}$ .

Answer (cont'd)

We now test each solution in the original inequality:

$$\begin{array}{ll} |x + 1| + |2x - 3| = 4 & \\ |3| + |1| = 4 & \text{True } x = 2 \\ |9| + |13| \neq 4 & \text{False } x = 8 \\ |1| + |-4| = 4 & \text{True } x = 0 \\ \left| \frac{2}{3} \right| + \left| -\frac{13}{3} \right| \neq 4 & \text{False } x = -\frac{2}{3} \end{array}$$

Therefore, the solutions are  $x = 0$  and  $x = 2$ .

## Outline

1 Inequalities involving the Absolute Value

2 Solving More General Inequalities

## Absolute Value &amp; Inequalities

## Question

Solve the inequality  $x^2 - 5x + 6 \leq 0$ .

## Answer

First we factor the left side:

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

We know that the corresponding equation has the solutions 2 and 3. The numbers 2 and 3 divide the real line into three intervals:

$$(-\infty, 2) \quad (2, 3) \quad (3, \infty)$$

On each of these intervals we determine the signs of the factors. For instance,

$$x \in (-\infty, 2) \Rightarrow x < 2 \Rightarrow x - 2 < 0$$

## Absolute Value &amp; Inequalities

$$x^2 - 5x + 6 \leq 0 \text{ (Cont'd)}$$

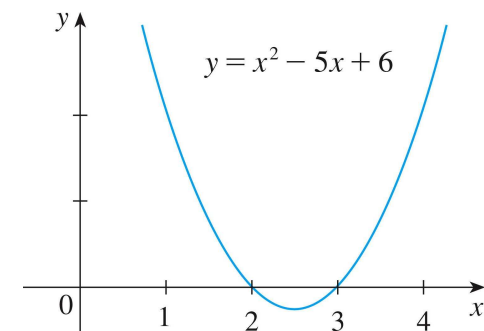
Then we record these signs in the following chart:

	$x - 2$	$x - 3$	$(x - 2)(x - 3)$
$(-\infty, 2)$	-	-	+
$(2, 3)$	+	-	-
$(3, \infty)$	+	+	+

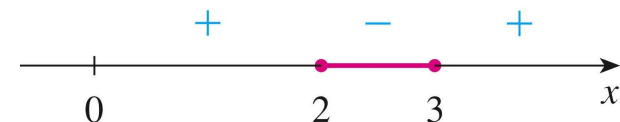
Therefore,  $x^2 - 5x - 6 \leq 0$  has solution

$$\{x \mid 2 \leq x \leq 3\} = [2, 3]$$

## Absolute Value &amp; Inequalities



$$x^2 - 5x - 6 \leq 0 \Rightarrow \{x \mid 2 \leq x \leq 3\} = [2, 3]$$



## Absolute Value & Inequalities

### Question

Solve the inequality  $\frac{(x+1)(x-3)}{(x+4)} > 0$ .

### Answer

Rather than attempt to sketch the function, we will first determine the points where it changes sign, i.e. those  $x$ -values which make the numerator or denominator zero:

$$x + 1 = 0 \Rightarrow x = -1$$

$$x - 3 = 0 \Rightarrow x = 3$$

$$x + 4 = 0 \Rightarrow x = -4$$

We now divide the  $x$ -axis into intervals according to these points, namely  $(-\infty, -4)$ ,  $(-4, -1)$ ,  $(-1, 3)$  and  $(3, \infty)$ .

$$\frac{(x+1)(x-3)}{(x+4)} > 0$$

### Answer (Cont'd)

We construct a table with these intervals and with the factors of the function as follows:

	$x + 1$	$x - 3$	$x + 4$	$f(x)$
$(-\infty, -4)$	-	-	-	-
$(-4, -1)$	-	-	+	+
$(-1, 3)$	+	-	+	-
$(3, \infty)$	+	+	+	+

We see that the function is positive on the two intervals  $(-4, -1)$  and  $(3, \infty)$ . We therefore write

$$\frac{(x+1)(x-3)}{x+4} > 0 \quad \text{on} \quad (-4, -1) \cup (3, \infty)$$